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PLANE ELECTROMAGNETIC WAVES

Wireless applications are possible because electromagnetic fields can propagate in free space without any guiding structures.

Plane waves are good approximations of electromagnetic waves in engineering problems after they propagate a short distance from the source.

Time-Harmonic Electromagnetics

In past chapter, field quantities are expressed as functions of time and position. In real applications, time signals can be expressed as superimposition of sinusoidal waveforms. So it is convenient to use the phasor notation to express fields in the frequency domain, just as in a.c. circuits.

$$\mathbf{E}(x,y,z,t) \equiv \mathrm{Re}\left\{\mathbf{E}(x,y,z)e^{j\omega t}\right\}$$

$$\frac{\partial \mathbf{E}(x,y,z,t)}{\partial t} = \mathrm{Re}\left\{\frac{\partial \mathbf{E}(x,y,z)e^{j\omega t}}{\partial t}\right\}$$

$$= \mathrm{Re}\left\{j\omega \mathbf{E}(x,y,z)e^{j\omega t}\right\}$$

Time-Harmonic Electromagnetics

Differential form of Maxwell's equations

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \cdot \mathbf{D} = \rho_v$$

$$\nabla \cdot \mathbf{B} = 0$$

Time-harmonic Maxwell's equations

$$\nabla \times \mathbf{E} = -j\omega \mathbf{B}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + j\omega \mathbf{D}$$

$$\nabla \cdot \mathbf{D} = \rho_v$$

$$\nabla \cdot \mathbf{B} = 0$$

Wave equations in Source-Free Media

In a source-free media (i.e.charge density ρ_v=0), Maxwell's equation becomes:

$$\nabla \times \mathbf{E} = -j\omega\mu \mathbf{H}$$

$$\nabla \times \mathbf{H} = (\sigma + j\omega\varepsilon)\mathbf{E}$$

$$\nabla \cdot \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{H} = 0$$

We will derive a differential equation involving **E** or **H** alone. First take the curl of both sides of the 1st equation:

$$\nabla \times \nabla \times \mathbf{E} = -j\omega\mu \nabla \times \mathbf{H}$$

$$\nabla \cdot \mathbf{E} - \nabla^2 \mathbf{E} = -j\omega\mu(\sigma + j\omega\varepsilon)\mathbf{E}$$

Since $\nabla \cdot \mathbf{E} = 0$ in source-free media, we obtain

$$\nabla^2 \mathbf{E} - \gamma^2 \mathbf{E} = 0$$

where

$$\gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon)$$

The above equation is called **Helmholtz equation**, and γ is called the propagation constant. In rectangular coordinates, Helmholtz equation can be decomposed into three scalar equations:

$$\nabla^2 E_i - \gamma^2 E_i = 0 \qquad i = x, y, z$$

Similarly for the magnetic field,

$$\nabla^2 \mathbf{H} - \gamma^2 \mathbf{H} = 0$$

Helmoltz Equation and Transmission Line equations

Helmoltz equation

$$\nabla^2 \mathbf{E} - \gamma^2 \mathbf{E} = 0$$

$$\nabla^2 \mathbf{H} - \gamma^2 \mathbf{H} = 0$$

Transmission Line equations

$$\frac{\partial^2 V}{\partial^2 z} = \gamma^2 V$$

$$\frac{\partial^2 I}{\partial^2 z} = \gamma^2 I$$

In rectangular co-ordinates

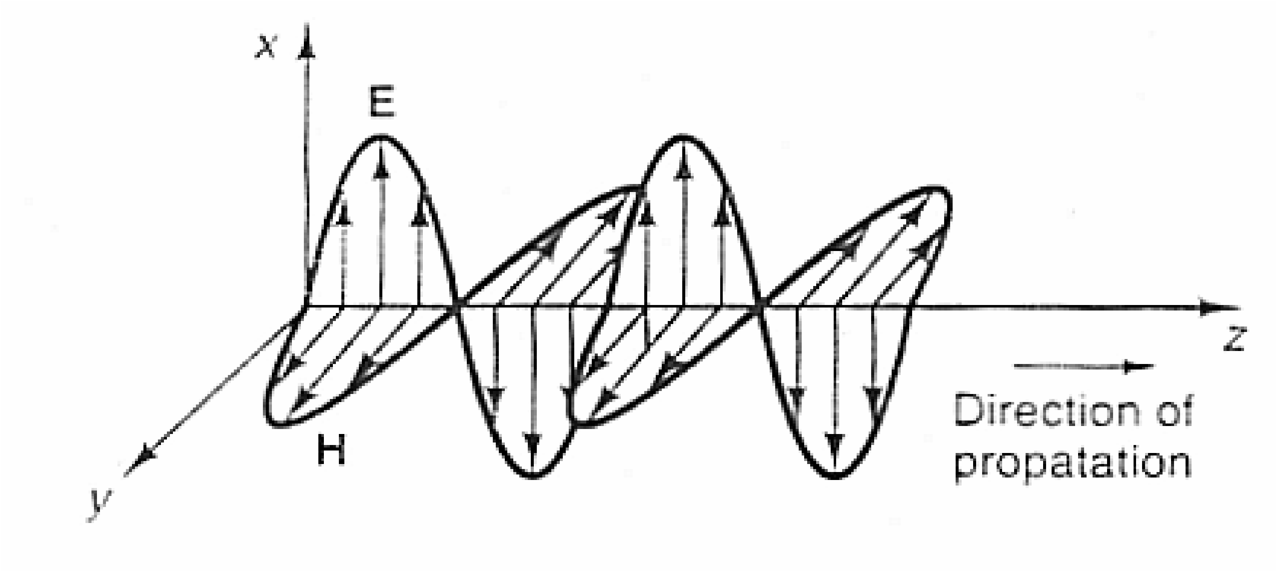
$$\nabla^2 E_i - \gamma^2 E_i = 0 \qquad i = x, y, z$$

Plane waves in lossless media

In a lossless medium, the conductive current is zero so σ is equal to zero; there is no energy loss and μ, ε are both real numbers. Therefore in lossless media

$$\gamma = j\omega\sqrt{\mu\varepsilon}$$

Plane wave is a particular solution of the Maxwell's equation, where both the electric field and magnetic field are perpendicular to the propagation direction of the wave. Let the wave propagate along the *z*-direction, and the electric field is along the *x*-direction.



E- and **H**-field vectors for a plane wave propagating in *z*-direction

$$\mathbf{E} = E_x(z)\mathbf{a_x}$$

Substituting this into Holmholtz equation, and since $\frac{\partial E_x}{\partial x} = \frac{\partial E_x}{\partial y} = 0$, we obtain

$$\frac{d^2 E_x}{dz^2} - \gamma^2 E_x = 0$$

This second order linear differential equation has two independent solutions, so the solution of E_x is

$$E_x(z) = E_{xo}^{+} e^{-\gamma z} + E_{xo}^{-} e^{+\gamma z}$$

The 1st term is a wave travelling in the +z direction, and the 2nd term is a wave travelling in the -z direction.

γ is called the propagation constant. In general, γ is complex

$$\gamma = \alpha + j\beta$$

where α, β are the attenuation and phase constants. But in lossless media,

$$\alpha = 0$$

$$\beta = \omega\sqrt{\mu\varepsilon} \qquad (\mu,\ \varepsilon \text{ are real})$$

So the electric field in lossless media is given by:

$$\mathbf{E} = \left(E_{xo}^{+} e^{-j\beta z} + E_{xo}^{-} e^{+j\beta z}\right)\mathbf{a}_{\mathbf{x}}$$

Let u_p be the phase velocity with which either the forward or backward wave is travelling, and λ is the wavelength.

Since $\beta = \omega\sqrt{\mu\varepsilon}$, therefore

$$u_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\varepsilon}}$$

$$\lambda = \frac{2\pi}{\beta}$$

In free space (vacuum), $\varepsilon = \varepsilon_o$, $\mu = \mu_o$ and the phase velocity is the same as the velocity of light in vacuum (3×10^8 m/s)

.

Similarly for the magnetic field, **H** can be found by solving

$$\mathbf{H} = \frac{\nabla \times \mathbf{E}}{-j\omega\mu}$$

under the conditions

$$\frac{\partial E_x}{\partial x} = \frac{\partial E_x}{\partial y} = 0$$

We obtain

$$\mathbf{H} = \frac{-1}{j\omega\mu}\frac{\partial}{\partial z}\left[E_{xo}^{+}e^{-j\beta z} + E_{xo}^{-}e^{+j\beta z}\right]\mathbf{a_y}$$

$$= \frac{\beta}{\omega\mu}\left[E_{xo}^{+}e^{-j\beta z} - E_{xo}^{-}e^{+j\beta z}\right]\mathbf{a_y}$$

We notice that the magnetic field is in the *y*-direction, i.e. perpendicular to the electric field, and the ratio of magnitudes of electric and magnetic fields is:

$$\frac{E_{xo}^{+}}{H_{xo}^{+}} = \frac{E_{xo}^{-}}{H_{xo}^{-}} = \eta$$

$$\eta = \frac{\omega\mu}{\beta} = \sqrt{\frac{\mu}{\varepsilon}}$$

η is called the intrinsic (or wave) impedance of the medium. In free space η has a value approximately equal to 377Ω.

The above plane wave is called **TEM** (transverse electromagnetic) wave, because both the electric and magnetic fields only exist in the transverse directions perpendicular to the propagation direction.

Other waves (e.g. in waveguides) can be also:

(a) **TE** (transverse electric) wave - electric field exists only in transverse direction, i.e. zero electric field in the propagation direction.

(b) **TM** (transverse magnetic) wave - magnetic field exists only in transverse direction, i.e. zero magnetic field in the propagation direction.

Example: A uniform plane wave with $\mathbf{E}=E_x\mathbf{a_x}$ propagates in a lossless medium ($\varepsilon_r=4$, $\mu_r=1$, $\sigma=0$) in the z-direction. If E_x has a frequency of 100MHz and has a maximum value of 10^{-4} (V/m) at $t=0$ and $z=1/8$ (m),

a) write the instantaneous expression for **E** and **H**,
b) determine the locations where E_x is a positive maximum when $t=10^{-8}$s.

Solution:

$$\gamma = j\omega\sqrt{\mu_o\mu_r\varepsilon_o\varepsilon_r} = \frac{j\omega}{c}\sqrt{\mu_r\varepsilon_r}$$

since velocity of light $\quad c = \dfrac{1}{\sqrt{\mu_o\varepsilon_o}} = 3\times10^8$

$$\gamma = j\frac{2\pi\times10^8}{3\times10^8}\sqrt{4} = \frac{4\pi}{3}j$$

(a) $$\mathbf{E} = \mathbf{a_x} 10^{-4} \cos\left(2\pi 10^8 t - \frac{4\pi}{3} z + \phi\right)$$

Setting E_x=10^{-4} at t=0 and z=1/8,

$$\phi = kz = \frac{\pi}{6}$$

The H-field: $$\mathbf{H} = \mathbf{a_y} H_y = \mathbf{a_y} \frac{E_x}{\eta}$$

$$\eta = \sqrt{\frac{\mu_r \mu_o}{\varepsilon_r \varepsilon_o}} = 60\pi$$

(b) At t=10^{-8}, for E_x to be maximum,

$$2\pi 10^8 (10^{-8}) - \frac{4\pi}{3}\left(z_m - \frac{1}{8}\right) = \pm 2n\pi$$

$$z_m = \frac{13}{8} \pm \frac{3}{2} n \qquad n = 0,1,2,....$$

7.4 Plane Waves in Lossy Media

The electric field in the *x*-direction is given by:

$$E_x(z) = E_{xo}^{+} e^{-\gamma z} + E_{xo}^{-} e^{+\gamma z}$$

where γ is in general complex for lossy media

$$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}$$

$$\gamma = \alpha + j\beta$$

So E-field can be written as:

$$\mathbf{E} = E_{xo}^{+} e^{-\alpha z} e^{-j\beta z} \mathbf{a_x} + E_{xo}^{-} e^{+\alpha z} e^{+j\beta z} \mathbf{a_x}$$

Substitute into Maxwell's equations, the magnetic field is:

$$\mathbf{H} = \frac{1}{\eta}\left[E_{xo}^{+} e^{-\gamma z} - E_{xo}^{-} e^{+\gamma z}\right]\mathbf{a}_{\mathbf{y}}$$

where the intrinsic impedance η is complex.

$$\eta = \frac{j\omega\mu}{\gamma} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}$$

The loss can be due to (i) conduction loss where σ is non-zero, (ii) dielectric loss where ε is complex,

$$\varepsilon = \varepsilon' - j\varepsilon''$$

Example: The E-field of a plane wave propagating in z-direction in sea water is $\mathbf{E}=\mathbf{a_x}100\cos(10^7\pi t)$ V/m at z=0. The parameters of sea water are ε_r=72, μ_r=1, σ=4 S/m. (a) Determine the attenuation constant, phase constant, intrinsic impedance,and phase velocity. (b) Find the distance where the amplitude of **E** is 1% of its value at z=0.

Solution: (a)

$$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}$$

$$= \alpha + j\beta$$

In our case $\sigma/\omega\varepsilon$=200>>1 so that α, β are approximately

$$\alpha = \beta = \sqrt{\pi f\mu\rho} = 8.89$$

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} \approx \sqrt{\frac{j\omega\mu}{\sigma}}$$

$$\eta = (1+j)\sqrt{\frac{\pi f\mu}{\sigma}} = (1+j)\sqrt{\frac{\pi(5\times10^{6})(4\pi\times10^{-7})}{4}} = \pi e^{j\pi/4}$$

Phase velocity: $u = \frac{\omega}{\beta} = \frac{10^{7}\pi}{8.89} = 3.53\times10^{6}$ m/s

(b) Distance z_1 where wave amplitude decreases to 1%:

$$e^{-\alpha z_1} = 0.01$$

$$z_1 = \frac{1}{\alpha}\ln 100 = \frac{4.605}{8.89} = 0.518$$

7.5 Polarization: Polarization describes how the **E**-field vector varies with time as the wave propagates. Let E_x, E_y be the components of **E**-field in the x and y directions.

• Linear polarization: The **E**-field vector is always in the same direction. E_x, E_y are either in phase or out of phase.

$$\mathbf{E}(t) = E_{m1}\cos(\omega t - \beta z)\mathbf{a}_\mathbf{x} + E_{m2}\cos(\omega t - \beta z)\mathbf{a}_\mathbf{y}$$

$$\mathbf{E}(t) = E_{m1}\cos(\omega t - \beta z)\mathbf{a}_\mathbf{x} - E_{m2}\cos(\omega t - \beta z)\mathbf{a}_\mathbf{y}$$

• Circular polarization: The **E**-field vector rotates around the axis of propagation and has constant amplitude. E_x, E_y have equal magnitudes and phase angle difference of $\pi/2$.

$$\mathbf{E}(t) = E_m \cos(\omega t - \beta z)\mathbf{a_x} + E_m \cos(\omega t - \beta z \pm \pi / 2)\mathbf{a_y}$$

• Elliptic polarization: The **E**-field vector rotates around the axis of propagation but has time-varying amplitude. E_x, E_y can have any values of magnitude and phase angle difference.

$$\mathbf{E}(t) = E_{m1} \cos(\omega t - \beta z)\mathbf{a_x} + E_{m2} \cos(\omega t - \beta z + \delta)\mathbf{a_y}$$

7.6 Power transmission - Poynting's Theorem

Power is transmitted by an electromagnetic wave in the direction of propagation. It can be derived from Maxwell's equation in the time domain that:

$$-\oint_S \mathbf{E} \times \mathbf{H} \cdot d\mathbf{S} = \int_V \mathbf{E} \cdot \mathbf{J} dv + \frac{\partial}{\partial t} \int_V \left(\frac{\varepsilon E^2}{2} + \frac{\mu H^2}{2} \right) dv$$

On the right hand side, the 1st term represents the instantaneous ohmic power loss, the 2nd and 3rd term represent the rate of increase of energies stored in the electric and magnetic fields respectively. So the left hand side represents the net instantaneous power supplied to the enclosed surface, or

$$W_{out} = \oint_S \mathbf{E} \times \mathbf{H}.d\mathbf{S}$$

$$\mathbf{P}(t) = \mathbf{E}(t) \times \mathbf{H}(t)$$

The instantaneous Poynting vector **P** (in watt per sq.m) represents the direction and density of power flow at a point.

In time-varying fields, it is more important to find the average power. We define the average Poynting vector for periodic signals as $\mathbf{P}_{\text{avg}}$

$$\mathbf{P}_{\mathbf{avg}} = \frac{1}{T}\int_0^T \mathbf{P}\,dt = \frac{1}{T}\int_0^T \mathbf{E}(t) \times \mathbf{H}(t)\,dt$$

Using complex phasor notation of **E, H,**

$$\mathbf{P}_{\mathbf{avg}} = \frac{1}{2}\mathrm{Re}\left(\mathbf{E} \times \mathbf{H}^*\right)$$

For a uniform plane wave, **E** & **H** field vectors are given by:

$$\mathbf{E} = E_{xo} e^{-\alpha z} e^{-j\beta z} \mathbf{a}_{\mathbf{x}}$$

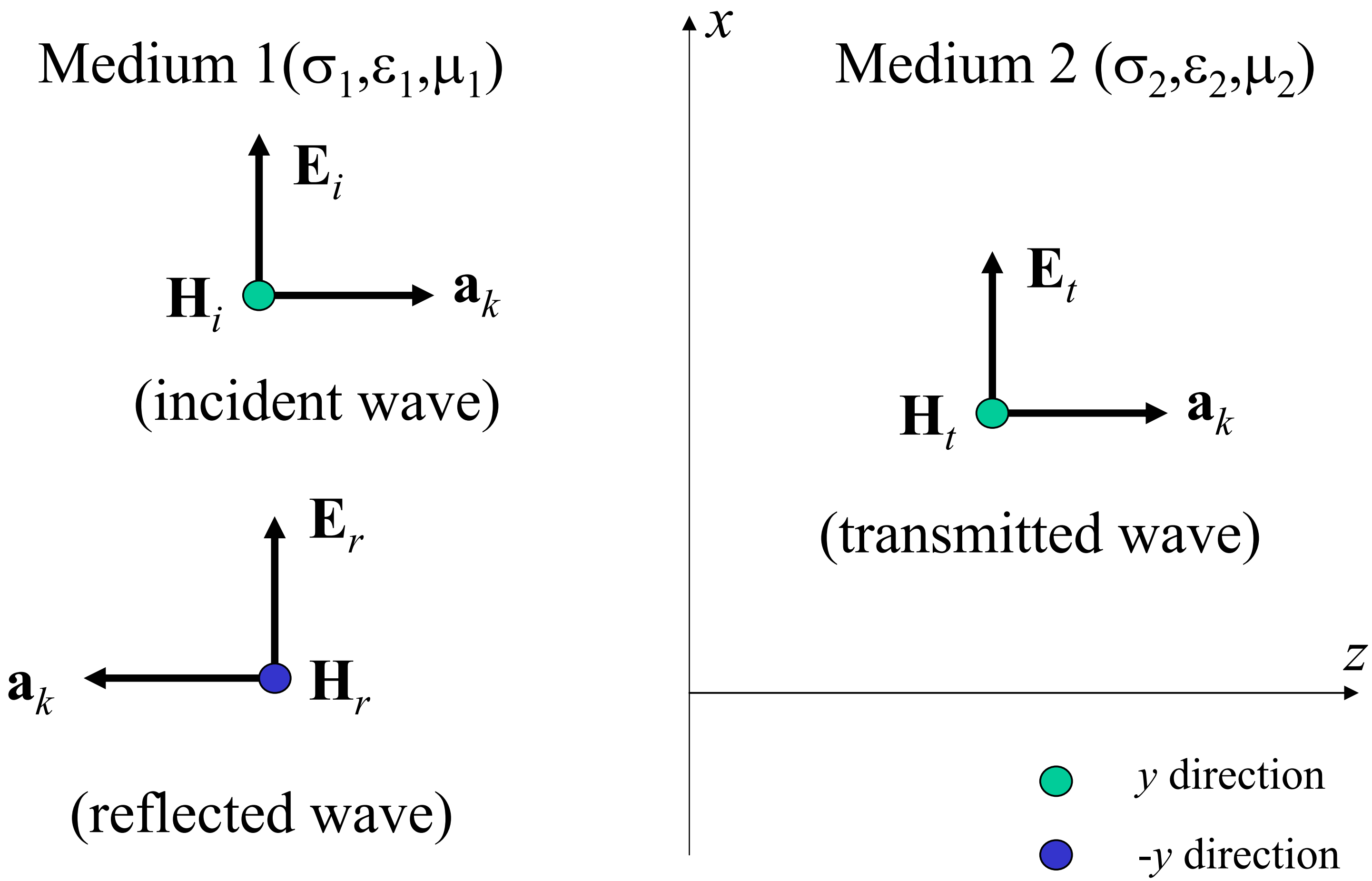
$$\mathbf{H} = \frac{1}{\eta}\left[E_{xo} e^{-\alpha z} e^{-j\beta z}\right]\mathbf{a}_{\mathbf{y}}$$

$$\eta = |\eta| e^{j\theta_\eta}$$

Substituting in equ. of $\mathbf{P}_{\mathbf{avg}}$ gives

$$\mathbf{P}_{\mathbf{avg}} = \frac{E_{xo}^2}{2|\eta|} e^{-2\alpha z} \cos\theta_\eta \mathbf{a}_{\mathbf{z}}$$

7.7 Reflection of Plane Waves at Normal Incidence



$\mathbf{a}_k$ – propagation direction

Incident wave:
$$\mathbf{E}_i = E_{io} e^{-\gamma_1 z} \mathbf{a}_x$$
$$\mathbf{H}_i = H_{io} e^{-\gamma_1 z} \mathbf{a}_y = \frac{E_{io}}{\eta_1} e^{-\gamma_1 z} \mathbf{a}_y$$

Reflected wave:
$$\mathbf{E}_r = E_{ro} e^{\gamma_1 z} \mathbf{a}_x$$
$$\mathbf{H}_r = H_{ro} e^{\gamma_1 z} (-\mathbf{a}_y) = -\frac{E_{ro}}{\eta_1} e^{\gamma_1 z} \mathbf{a}_y$$

Transmitted wave:
$$\mathbf{E}_t = E_{to} e^{-\gamma_2 z} \mathbf{a}_x$$
$$\mathbf{H}_t = H_{to} e^{-\gamma_2 z} \mathbf{a}_y = \frac{E_{to}}{\eta_2} e^{-\gamma_2 z} \mathbf{a}_y$$

Boundary conditions at the interface (z=0):

Tangential components of **E** and **H** are continuous in the absence of current sources at the interface (refer to tutorials), so that

$$E_{io} + E_{ro} = E_{to}$$

$$H_{io} + H_{ro} = H_{to}$$

$$\frac{1}{\eta_1}\left(E_{io} - E_{ro}\right) = \frac{E_{to}}{\eta_2}$$

Reflection coefficient $\Gamma = \frac{E_{ro}}{E_{io}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$

Transmission coefficient $\tau = \frac{E_{to}}{E_{io}} = \frac{2\eta_2}{\eta_1 + \eta_2}$

Note that Γ and τ may be complex, and

$$1 + \Gamma = \tau$$

$$0 \leq |\Gamma| \leq 1$$

Similar expressions may be derived for the magnetic field.

In medium 1, a standing wave is formed due to the superimposition of the incident and reflected waves. Standing wave ratio can be defined as in transmission lines.

END